

Real Options and Threshold Strategies

Vadim Arkin and Alexander Slastnikov

Central Economics and Mathematics Institute, Russian Academy of Sciences,
Nakhimovskii pr. 47, 117418 Moscow, Russia

Abstract. The paper considers an investment timing problem appearing in real options theory. Present values from an investment project are modeled by general diffusion process. We prove necessary and sufficient conditions under which an optimal investment time is induced by threshold strategy. We study also the conditions of optimality of threshold strategy (over all threshold strategies) and discuss the connection between solutions to investment timing problem and to free-boundary problem.

Keywords: real options, investment timing problem, diffusion process, optimal stopping, threshold stopping time, free-boundary problem

1 Introduction

One of the fundamental problems in real options theory concerns the determination of optimal time for investment into a given project (see, e.g., classical monograph [6]).

Let us consider an investment project, for example, a creating of new firm in the real sector of economy. This project is characterized by a pair $(X_t, t \geq 0, I)$, where X_t is a present value of the firm created at time t , and I is a cost of investment required to implement the project (for example, to create the firm). Prices on input and output production are assumed to be stochastic, so X_t is considered as a stochastic process, defined at a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$. This model supposes that:

- at any moment, a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information;
- investment are considered to be instantaneous and irreversible so that they cannot be withdrawn from the project any more and used for other purposes.

The investor's problem is to evaluate the project and to determine an appropriate time for the investment (investment timing problem). In real option theory investment times are considered as stopping times (regarding to flow of σ -algebras $\{\mathcal{F}_t, t \geq 0\}$).

In real options theory there are two different approaches to solving investment timing problem (see [6]).

The value of project under the first approach is the maximum of net present value (NPV) from the implemented project over all stopping times (investment rules):

$$F = \max_{\tau} \mathbf{E}(X_{\tau} - I)e^{-\rho\tau}, \quad (1)$$

where ρ is the given discount rate. An optimal stopping time τ^* in (1) is viewed as optimal investment time (investment rule).

Within the second approach an opportunity to invest is considered as an American call option – the right but not obligation to buy the asset on predetermined price. At that an exercise time is viewed as investment time, and value of option is accepted as a value of investment project. In these framework a project is spanned with some traded asset, which price is completely correlated with present value of the project X_t . In order to evaluate a (rational) value of this real option one can use methods of financial options pricing theory, especially, contingent claims analysis (see, e.g., [6]).

In this paper we follow the approach that optimal investment timing decision can be mathematically determined as a solution of an optimal stopping problem (1). Such an approach started from the well-known McDonald–Siegel model (see [11], [6]), in which the underlying present value's dynamics is modeled by a geometric Brownian motion. The majority of results on this problem (optimal investment strategy) has a threshold structure: to invest when present value from the project exceeds the certain level (threshold). In the heuristic level this is so for the cases of geometric Brownian motion, arithmetic Brownian motion, mean-reverting process and some other (see [6]). And the general question arises: For what underlying processes an optimal decision to an investment timing problem will have a threshold structure?

Some sufficient conditions in this direction was obtained in [1]. In this paper we focus on necessary and sufficient conditions for optimality of threshold strategies in investment timing problem. Since this problem is a special case of optimal stopping problem, the similar question may be addressed to a general optimal stopping problem: Under what conditions (on both process and payoff function) an optimal stopping time will have a threshold structure? Some results in this direction (in the form of necessary and sufficient conditions) were obtained in [3], [5], [2] under some additional assumptions on underlying process and/or payoffs.

The paper is organized as follows. After a formal description of investment timing problem and assumptions on underlying process (Section 2.1) we go to study of threshold strategies for this problem. Since an investment timing problem in threshold strategies is reduced to one-dimensional maximization, a related problem is to find an optimal threshold. In Section 2.2 we give necessary and sufficient conditions for optimal threshold (over all thresholds). Solving a free-boundary problem (based on smooth-pasting principle) is the most accepted method (but not the only method, see, e.g., [12]) that allows to find a solution to optimal stopping problem. In Section 2.3 we discuss the connection between solutions to investment timing problem and to free-boundary problem. Finally,

in Section 2.4 we prove the main result on necessary and sufficient conditions under which an optimal investment time is generated by threshold strategy.

2 Investment Timing Problem

Let I be a cost of investment required for implementing a project, and X_t is the present value from the project started at time t . As usual investment supposed to be instantaneous and irreversible, and the project — infinitely-lived.

At any time a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until he/she obtains new information regarding its environment (prices of the product and resources, demand etc.). The goal of a decision-maker in this situation is to find, using the available information, an optimal time for investing the project (investment timing problem), which maximizes the net present value from the project:

$$\mathbf{E}^x (X_\tau - I) e^{-\rho\tau} \mathbf{1}_{\{\tau < \infty\}} \rightarrow \max_{\tau \in \mathcal{M}}, \quad (2)$$

where \mathbf{E}^x means the expectation for the process X_t starting from the initial state x , $\mathbf{1}_A$ is indicator function of the set A , and the maximum is considered over stopping times τ from a certain class \mathcal{M} of stopping times¹.

We consider the case $I < r$ else the optimal time in (2) will be $+\infty$.

2.1 Mathematical Assumptions

Let X_t be a diffusion process with values in the interval $D \subseteq \mathbb{R}^1$ with boundary points l and r , where $-\infty \leq l < r \leq +\infty$, open or closed (i.e. it may be (l, r) , $[l, r)$, $(l, r]$, or $[l, r]$), which is a solution to stochastic differential equation:

$$dX_t = a(X_t)dt + \sigma(X_t)dw_t, \quad X_0 = x, \quad (3)$$

where w_t is a standard Wiener process, $a : D \rightarrow \mathbb{R}^1$ and $\sigma : D \rightarrow \mathbb{R}_+^1$ are the drift and diffusion functions, respectively. Denote $\mathcal{I} = \text{int}(D) = (l, r)$.

The process X_t is assumed to be regular; this means that, starting from an arbitrary point $x \in \mathcal{I}$, this process reaches any point $y \in \mathcal{I}$ in finite time with positive probability.

It is known that the following local integrability condition:

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |a(y)|}{\sigma^2(y)} dy < \infty \quad \text{for some } \varepsilon > 0, \quad (4)$$

at any $x \in \mathcal{I}$ guarantees the existence of weak solution of equation (3) and its regularity (see, e.g. [10]).

¹ In this paper we consider stopping times which can take infinite values (with positive probability)

The process X_t is associated with infinitesimal operator

$$\mathbb{L}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x). \quad (5)$$

Under the condition (4) there exist (unique up to constant positive multipliers) increasing and decreasing functions $\psi(x)$ and $\varphi(x)$ with absolutely continuous derivatives, which are the fundamental solutions to the ODE

$$\mathbb{L}f(p) = \rho f(p) \quad (6)$$

almost sure (in Lebesgue measure) on the interval \mathcal{I} (see, e.g. [10, Chapter 5, Lemma 5.26]). Moreover, $0 < \psi(p), \varphi(p) < \infty$ for $p \in \mathcal{I}$. Note, if functions $a(x)$, $\sigma(x)$ are continuous, then $\psi, \varphi \in C^2(\mathcal{I})$.

2.2 Optimality of Threshold Strategies

Let us define $\tau_p = \tau_p(x) = \inf\{t \geq 0 : X_t \geq p\}$ — the first time when the process X_t , starting from x , exceeds level p . We will call τ_p as threshold stopping time generated by the threshold strategy — to stop when the process exceeds threshold p . Let $\mathcal{M}_{\text{th}} = \{\tau_p, p \in \mathcal{I}\}$ be a class of all such threshold stopping times.

For the class \mathcal{M}_{th} of threshold stopping times the investment timing problem (2) can be written as follows:

$$(p - I) \mathbf{E}^x e^{-\rho\tau_p} \rightarrow \max_{p \in (l, r)}. \quad (7)$$

Such a problem appeared in [7] as the heuristic method for solving a general investment timing problem (2) over class of all stopping times.

We say that threshold p^* is optimal for the investment timing problem (7) if threshold stopping time τ_{p^*} is optimal in (7). The following result gives necessary and sufficient conditions for optimal threshold.

Theorem 1. *Threshold $p^* \in \mathcal{I}$ is optimal in the problem (7) for all $x \in \mathcal{I}$, if and only if the following conditions hold:*

$$\frac{p - I}{\psi(p)} \leq \frac{p^* - I}{\psi(p^*)} \quad \text{whenever } p < p^*; \quad (8)$$

$$\frac{p - I}{\psi(p)} \quad \text{does not increase for } p \geq p^*, \quad (9)$$

where $\psi(p)$ is an increasing solution to ODE (6).

Proof. Let us denote the left-hand side in (7) as $V(p; x)$. Obviously, $V(p; x) = x - I$ for $x \geq p$.

Along with the above stopping time let us define the first hitting time to threshold: $T_p = \inf\{t \geq 0 : X_t = p\}$, $p \in (l, r)$.

For $x < p$, obviously, $\tau_p = T_p$ and using known formula $\mathbf{E}^x e^{-\rho T_p} = \psi(x)/\psi(p)$ (see, e.g., [9], [4]) we have:

$$V(p; x) = (p - I)\mathbf{E}^x e^{-\rho \tau_p} \mathbf{1}_{\{\tau_p < \infty\}} = (p - I)\mathbf{E}^x e^{-\rho T_p} = \frac{p - I}{\psi(p)} \psi(x). \quad (10)$$

Denote $h(x) = (x - I)/\psi(x)$.

i) Let $p^* \in \mathcal{I}$ be an optimal threshold in the problem (7) for all $x \in \mathcal{I}$. Then for $p < p^*$ we have

$$V(p; p) = p - I \leq V(p^*; p) = \frac{p^* - I}{\psi(p^*)} \psi(p),$$

i.e. (8) holds. If $p^* \leq p_1 < p_2$, then

$$V(p_2; p_1) = h(p_2)\psi(p_1) \leq V(p^*; p_1) = p_1 - I = h(p_1)\psi(p_1),$$

and it follows (9).

ii) Now, let (8)–(9) hold.

Let $p < p^*$. If $x \geq p^*$, then $V(p; x) = x - I = V(p^*; x)$.

If $p \leq x < p^*$, then, due to (8), $V_p(x) = x - I = h(x)\psi(x) \leq h(p^*)\psi(x) = V(p^*; x)$.

Finally, if $x < p$, then, using (8) and (10), we have: $V(p; x) = h(p)\psi(x) \leq h(p^*)\psi(x) = V(p^*; x)$.

Consider the case $p > p^*$. If $x \geq p$, then $V(p; x) = x - I = V(p^*; x)$.

Whenever $p^* \leq x < p$, then, due to (9), $V(p; x) = h(p)\psi(x) \leq h(x)\psi(x) = x - I = V(p^*; x)$.

When $x < p^*$, then $V(p; x) = h(p)\psi(x) \leq h(p^*)\psi(x) = V(p^*; x)$, since $h(p) \leq h(p^*)$.

Theorem is completely proved.

Remark 1. The condition (9) is equivalent to the inequality

$$(p - I)\psi'(p) \geq \psi(p) \quad \text{for } p \geq p^*.$$

This relation implies, in particular, that optimal threshold p^* must be strictly greater than investment cost I (because $\psi(p^*)$, $\psi'(p^*)$ are positive values).

Remark 2. Assume that $\log \psi(x)$ is a convex function, i.e. $\psi'(x)/\psi(x)$ increases. For this case there exists a unique point p^* which satisfies the equation

$$(p^* - I)\psi'(p^*) = \psi(p^*) \quad (11)$$

and constitutes the optimal threshold in the problem (7) for all $x \in \mathcal{I}$. Indeed, the sign of derivative of the function $(p - I)/\psi(p)$ coincides with the sign of $\psi(p) - (p - I)\psi'(p)$. Therefore, in the considered case the conditions (8)–(9) in Theorem 1 are true automatically.

We can give a number of cases of diffusion processes which are more or less realistic for modeling a process X_t of present values from a project. Some of them are presented below.

1) *Geometric Brownian motion (GBM)*:

$$dX_t = X_t(\alpha dt + \sigma dw_t). \quad (12)$$

For this case $\psi(x) = x^\beta$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - \rho = 0$.

2) *Arithmetic Brownian motion (ABM)*:

$$dX_t = x + \alpha dt + \sigma dw_t. \quad (13)$$

For this case $\psi(x) = e^{\beta x}$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta^2 + \alpha\beta - \rho = 0$.

3) *Mean-reverting process (or geometric Ornstein-Uhlenbeck process)*:

$$dX_t = \alpha(\bar{x} - X_t)X_t dt + \sigma X_t dw_t. \quad (14)$$

For this case $\psi(x) = x^\beta {}_1F_1\left(\beta, 2\beta + \frac{2\alpha\bar{x}}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)$, where β is the positive root of equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\bar{x}\beta - \rho = 0$, and ${}_1F_1(p, q; x)$ is confluent hypergeometric function satisfying Kummer's equation $xf''(x) + (q-x)f'(x) - pf(x) = 0$.

4) *Square-root mean-reverting process (or Cox-Ingersoll-Ross process)*:

$$dX_t = \alpha(\bar{x} - X_t)dt + \sigma\sqrt{X_t}dw_t. \quad (15)$$

For this case $\psi(x) = {}_1F_1\left(\frac{\rho}{\alpha}, \frac{2\alpha\bar{x}}{\sigma^2}; \frac{2\alpha}{\sigma^2}x\right)$.

The above processes are well studied in the literature (in connection with real options and optimal stopping problems see, for example, [6], [8]).

For the first two processes (12) and (13) the conditions of Theorem 1 give explicit formulas for optimal threshold in investment timing problem:

$$p^* = \frac{\beta}{\beta-1}I \text{ for GBM, and } p^* = I + \frac{1}{\beta} \text{ for ABM.}$$

On the contrary, for mean-reverting processes (14) and (15) the function $\psi(x)$ is represented as infinite series, and optimal threshold can be find only numerically.

So, Theorem 1 states that optimal threshold p^* is a point of maximum for the function $h(x) = (x - I)/\psi(x)$. This implies the first-order optimality condition $h'(p^*) = 0$, i.e. the equality (11), and smooth-pasting principle:

$$V'_x(p^*; x)|_{x=p^*} = 1.$$

In the next section we discuss smooth-pasting principle and appropriate free-boundary problem more closely.

2.3 Threshold Strategies and Free-Boundary Problem

It is almost common opinion (especially among engineers and economists) that solution to free-boundary problem always gives a solution to optimal stopping problem.

A free-boundary problem for the case of threshold strategies in investment timing problem can be written as follows: to find threshold $p^* \in (l, r)$ and twice differentiable function $H(x)$, $l < x < p^*$, such that

$$\mathbb{L}H(x) = \rho H(x), \quad l < x < p^*; \quad (16)$$

$$H(p^* - 0) = p^* - I, \quad H'(p^* - 0) = 1. \quad (17)$$

If $\psi(x)$ is twice differentiable, then solution to the problem (16)–(17) has the type

$$H(x) = \frac{p^* - I}{\psi(p^*)} \psi(x), \quad l < x < p^*, \quad (18)$$

where $\psi(x)$ is an increasing solution to ODE (6) and p^* satisfies the smooth-pasting condition (11). In further, we will call such p^* the solution to free-boundary problem.

According to Theorem 1 the optimal threshold in problem (7) must be a point of maximum of the function $h(x) = (x - I)/\psi(x)$, but smooth-pasting condition (11) provides only a stationary point for $h(x)$. Thus, we can apply standard second-order optimality conditions to derive relations between solutions to investment timing problem and to free-boundary problem.

Let p^* be a solution to free-boundary problem (16)–(17). If p^* is also an optimal threshold in investment timing problem (7), then, of course, $h''(p^*) \leq 0$. It means that

$$\psi''(p^*) = -\frac{h''(p^*)\psi(p^*) + 2h'(p^*)\psi'(p^*)}{h(p^*)} = -\frac{h''(p^*)\psi(p^*)}{h(p^*)} \geq 0.$$

Thus, the inequality $\psi''(p^*) \geq 0$ may be viewed as necessary condition for a solution of free-boundary problem to be optimal in investment timing problem.

The inverse relation between solutions can be state as follows.

Statement 1. *If p^* is the unique solution to free-boundary problem (16)–(17), and $\psi''(p^*) > 0$, then p^* is optimal threshold in the problem (7) for all $x \in \mathcal{I}$.*

Proof. Since $h'(p^*) = 0$ and $\psi''(p^*) > 0$ then $h''(p^*) = -h(p^*)\psi''(p^*)/\psi(p^*) < 0$. Therefore, $h'(p)$ strictly decreases at some neighborhood of p^* .

Then, it is easy to see that $h'(p) > 0$ for $p < p^*$ and $h'(p) < 0$ for $p > p^*$, else $h'(q) = 0$ for some $q \neq p^*$, that contradicts to the uniqueness of solution to free-boundary problem (16)–(17). So, conditions (8)–(9) hold and Theorem 1 gives the optimality of threshold p^* .

The following result concerns the general case when free-boundary problem has several solutions.

Statement 2. Let p^* and \tilde{p} are two solutions to free-boundary problem (16)–(17), such that $\psi''(p^*) > 0$ and $(x - I)/\psi(x) \leq (p^* - I)/\psi(p^*)$ for $l < x < p^*$. If $\tilde{p} > p^*$, and $\psi^{(k)}(\tilde{p}) = 0$ ($k = 2, \dots, n - 1$), $\psi^{(n)}(\tilde{p}) > 0$ for some $n > 2$, then p^* is optimal threshold in the problem (7) for all $x \in \mathcal{I}$.

Proof. Let us prove that $h'(p) \leq 0$ for all $p > p^*$. Inequality $\psi''(p^*) > 0$ implies (as above) that $h''(p^*) < 0$, and, therefore, $h'(p) < 0$ for all $p^* < p < p_1$ with some p_1 . If we suppose that $h'(p_2) > 0$ for some $p_2 > p^*$, then there exists $p_0 \in (p_1, p_2)$ such that $h'(p_0) = 0$ and $h'(p) > 0$ for all $p_0 < p < p_2$. Therefore, p_0 is another solution to free-boundary problem (16)–(17), and due to conditions of the Statement $h^{(k)}(p_0) = 0$ ($k = 2, \dots, n - 1$), $h^{(n)}(p_0) < 0$ for some $n > 2$, that contradicts to positivity of $h'(p)$ for $p_0 < p < p_2$.

Hence, $h'(p) \leq 0$ for all $p > p^*$ and conditions (8)–(9) hold. Thus, according to Theorem 1, p^* is optimal threshold in the problem (7).

2.4 Optimal Strategies in Investment Timing Problem

Now, return to ‘general’ investment timing problem (2) over the class \mathcal{M} of all stopping times.

The sufficient conditions under which an optimal investment time in (2) will be a threshold stopping time were derived in [1]. In this section we give necessary and sufficient conditions (criterion) for optimality of threshold stopping time in investment timing problem (2).

To reduce some technical difficulties we assume below that drift $a(x)$ and diffusion $\sigma(x)$ of the underlying process X_t are continuous functions.

Theorem 2. Threshold stopping time τ_{p^*} , $p^* \in (l, r)$, is optimal in the investment timing problem (2) for all $x \in \mathcal{I}$ if and only if the following conditions hold:

$$(p - I)\psi(p^*) \leq (p^* - I)\psi(p) \quad \text{for } p < p^*; \quad (19)$$

$$\psi(p^*) = (p^* - I)\psi'(p^*); \quad (20)$$

$$a(p) \leq \rho(p - I) \quad \text{for } p > p^*, \quad (21)$$

where $\psi(x)$ is an increasing solution to ODE (6) and $a(p)$ is the drift function of the process X_t .

Proof. Define the value function for the problem (2) over the class \mathcal{M} of all stopping times

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}^x (X_\tau - I) e^{-\rho\tau} \mathbf{1}_{\{\tau < \infty\}}.$$

i) Let conditions (19)–(21) hold.

Take the function

$$\Phi(x) = V(p^*; x) = \begin{cases} \frac{p^* - I}{\psi(p^*)} \psi(x), & \text{for } x < p^*, \\ x - I, & \text{for } x \geq p^*. \end{cases}$$

Obviously, $\Phi(x) > 0$ (due to condition (20)) and $V(x) \geq \Phi(x)$.
On the other hand, (19) implies

$$\frac{p^* - I}{\psi(p^*)} \psi(x) \geq \frac{x - I}{\psi(x)} \psi(x) = x - I,$$

therefore $\Phi(x) \geq x - I$ for all $x \in (l, r)$, i.e. $\Phi(x)$ is a majorant of payoff function $x - I$.

For any stopping time $\tau \in \mathcal{M}$ and $N > 0$ put $\tilde{\tau} = \tau \wedge N$. From Itô–Tanaka–Meyer formula (see, e.g. [10]) we have:

$$\begin{aligned} \mathbf{E}^x \Phi(X_{\tilde{\tau}}) e^{-\rho \tilde{\tau}} &= \Phi(x) + \mathbf{E}^x \int_0^{\tilde{\tau}} (\mathbb{L}\Phi - \rho\Phi)(X_t) e^{-\rho t} dt \\ &\quad + \frac{1}{2} \sigma^2(p^*) [\Phi'(p^*+0) - \Phi'(p^*-0)] \mathbf{E}^x \int_0^{\tilde{\tau}} e^{-\rho t} dL_t(p^*), \end{aligned} \quad (22)$$

where $L_t(p^*)$ is the local time of the process X_t at the point p^* .

By definition we have

$$\Phi'(p^*+0) - \Phi'(p^*-0) = 1 - \frac{p^* - I}{\psi(p^*)} \psi'(p^*) = 0$$

due to (20).

Take $T_1 = \{0 \leq t \leq \tilde{\tau} : X_t < p^*\}$, $T_2 = \{0 \leq t \leq \tilde{\tau} : X_t > p^*\}$. We have:

$$\begin{aligned} \mathbb{L}\Phi(X_t) - \rho\Phi(X_t) &= \frac{p^* - I}{\psi(p^*)} (\mathbb{L}\psi(X_t) - \rho\psi(X_t)) = 0 \quad \text{for } t \in T_1, \\ \mathbb{L}\Phi(X_t) - \rho\Phi(X_t) &= a(X_t) - \rho(X_t - I) \leq 0 \quad \text{for } t \in T_2 \end{aligned}$$

by definition of the function $\psi(x)$ and (21)).

Then

$$\begin{aligned} \mathbf{E}^x \Phi(X_{\tilde{\tau}}) e^{-\rho \tilde{\tau}} &\leq \Phi(x) + \mathbf{E}^x \left(\int_{T_1} (\mathbb{L}\Phi - \rho\Phi)(X_t) e^{-\rho t} dt + \int_{T_2} (\mathbb{L}\Phi - \rho\Phi)(X_t) e^{-\rho t} dt \right) \\ &\leq \Phi(x). \end{aligned}$$

Since $\Phi(X_{\tilde{\tau}}) e^{-\rho \tilde{\tau}} \xrightarrow{\text{a.s.}} \Phi(X_{\tau}) e^{-\rho \tau} \mathbf{1}_{\{\tau < \infty\}}$ when $N \rightarrow \infty$, then due to Fatou's Lemma : $\mathbf{E}^x \Phi(X_{\tau}) e^{-\rho \tau} \mathbf{1}_{\{\tau < \infty\}} \leq \Phi(x)$ for all $\tau \in \mathcal{M}$ and $x \in \mathcal{I}$. Therefore, $\Phi(x)$ is ρ -excessive function, which majorates payoff function $x - I$. Since, by Dynkin's characterization, value function $V(x)$ is the least ρ -excessive majorant, then $V(x) \leq \Phi(x)$.

Therefore, $V(x) = \Phi(x) = V(p^*; x)$, i.e. τ_{p^*} is the optimal stopping time in problem (2) for all x .

ii) Now, let τ_{p^*} be optimal stopping time in the problem (2). Note, that τ_{p^*} will be an optimal stopping time in the problem (7) also. Therefore, Theorem 1 implies (19) and (20), since p^* is point of maximum for the function $(x - I)/\psi(x)$.

Further, assume that inequality (21) is not true at some point $p_0 > p^*$, i.e. $a(p) > \rho(p - I)$ in some interval $J \subset (p^*, r)$ (by virtue of continuity). For some $\tilde{x} \in J$ define $\tau = \inf\{t \geq 0 : X_t \notin J\}$, where process X_t starts from the point \tilde{x} . Then for any $N > 0$ from Dynkin's formula

$$\mathbf{E}^{\tilde{x}}(X_{\tau \wedge N} - I)e^{-\rho(\tau \wedge N)} = \tilde{x} - I + \mathbf{E}^{\tilde{x}} \int_0^{\tau \wedge N} [a(X_t) - \rho(X_t - I)]e^{-\rho t} dt > \tilde{x} - I.$$

Therefore, $V(\tilde{x}) > \tilde{x} - I$ that contradicts to $V(\tilde{x}) = V(p^*; \tilde{x}) = g(\tilde{x})$, since $\tilde{x} > p^*$.

Example 1. Let X_t be the process of geometric Brownian motion (12). Then Theorem 2 implies that threshold stopping time τ_{p^*} will be optimal in the investment timing problem (2) over all investment times if and only if $p^* = I\beta/(\beta - 1)$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - \rho = 0$.

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